Please check the examination details below before entering your candidate information

Candidate surname	Other names

Centre Number | Candidate Number

Pearson Edexcel Level 3 GCE

Thursday 25 May 2023

Afternoon
(Time: 1 hour 30 minutes)

Paper reference **9FM0/01**

Further Mathematics

Advanced

PAPER 1: Core Pure Mathematics 1

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided
  – there may be more space than you need.
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets
  – use this as a guide as to how much time to spend on each question.

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

P 7 2 7 9 4 A 0 1 2 8

Pearson

1. The cubic equation

$$x^3 - 7x^2 - 12x + 6 = 0$$

has roots α, β and γ.

Without solving the equation, determine a cubic equation whose roots are $(\alpha + 2)$, $(\beta + 2)$ and $(\gamma + 2)$, giving your answer in the form $w^3 + pw^2 + qw + r = 0$, where p, q and r are integers to be found.

(5)

METHOD 1: linear transformation substitution

remembering the substitution to use to get the transformed equation given its transformed points: let $w = x+2$

solve for x

$\Rightarrow x = w-2$

and sub into the original equation (to get equation in terms of 'w')

$(w-2)^3 - 7(w+2)^2 - 12(w-2) + 6 = 0$

...using BINOMIAL EXPANSION:

PASCAL'S TRIANGLE

1
1 1
1 2 1
1 3 3 1

• $(w-2)^3$

$= (w)^3 + 3(w^2)(-2) + 3(w)(-2)^2 + (-2)^3$

$= w^3 - 6w^2 + 12w - 8$

• $-7(w-2)^2$

$= -7(w^2 - 4w + 4)$

$= -7w^2 + 28w - 28$

• $-12(w-2)$

$= -12w + 24$

• $+6$

Subbing into above:

$w^3 - 6w^2 + 12w - 8 - 7w^2 + 28w - 28 - 12w + 24 + 6$

collect like terms

$w^3 + (-6-7)w^2 + (12+28-12)w - 6 = 0$

$\Rightarrow \boxed{w^3 - 13w^2 + 28w - 6 = 0}$

METHOD 2: using the transformed roots of polynomials formulae

first for original

$$\underset{\Sigma\alpha}{x^3 - 7x^2} \underset{\Sigma\alpha\beta}{- 12x} \underset{\alpha\beta\gamma}{+ 6} = 0$$

• $\Sigma\alpha = -b/a$
$= 7$

• $\Sigma\alpha\beta = c/a$
$= -12$

• $\alpha\beta\gamma = -d/a$
$= -6$

P 7 2 7 9 4 A 0 2 2 8

Question 1 continued

now for transformed:

$$w^3+\underset{\Sigma\alpha}{p}w^2+\underset{\Sigma\alpha\beta}{q}w+\underset{\alpha\beta\gamma}{r}=0$$

$\Sigma\alpha=-p$

$\Sigma\alpha\beta=q$

$\alpha\beta\gamma=-r$

$\Sigma\alpha = \alpha+2+\beta+2+\gamma+2$

$-p = \Sigma\alpha+6$

$-p = 7+6$

$\Rightarrow -p=13$ ÷-1

÷-1 $\Rightarrow p=-13$

$\Sigma\alpha\beta = (\alpha+2)(\beta+2)+(\alpha+2)(\gamma+2)+(\beta+2)(\gamma+2)$

expand

$q = \alpha\beta+2\alpha+2\beta+4+\alpha\gamma+2\alpha+2\gamma+4+\beta\gamma+2\beta+2\gamma+4$

$= 4(\Sigma\alpha)+\Sigma\alpha\beta+12$

$= 4(7)+(-12)+12$

$\Rightarrow q = 28$

$\alpha\beta\gamma = (\alpha+2)(\beta+2)(\gamma+2)$

$-r = (\alpha\beta+2\alpha+2\beta+4)(\gamma+2)$

expand

$= \alpha\beta\gamma+2\alpha\gamma+2\beta\gamma+4\gamma+2\alpha\beta+4\alpha+4\beta+8$

$= 4(\Sigma\alpha)+2(\Sigma\alpha\beta)+\alpha\beta\gamma+8$

$= 4(7)+2(-12)+(-6)+8$

$= 28-24-6+8$

$-r=6$

$\Rightarrow r=-6$

$\therefore$ subbing into transformed equation:

$$\boxed{w^3-13w^2+28w-6=0}$$

(Total for Question 1 is 5 marks)

P 7 2 7 9 4 A 0 3 2 8

2. (a) Write $x^2 + 4x - 5$ in the form $(x + p)^2 + q$ where p and q are integers.

(1)

(b) Hence use a standard integral from the formula book to find

$$\int \frac{1}{\sqrt{x^2 + 4x - 5}}\,dx$$

(2)

(c) Determine the mean value of the function

$$f(x) = \frac{1}{\sqrt{x^2 + 4x - 5}} \qquad 3 \leqslant x \leqslant 13$$

giving your answer in the form $A \ln B$ where A and B are constants in simplest form.

(3)

(a) complete the square for given quadratic

$$x^2 + 4x - 5 = (x+2)^2 - 2^2 - 5$$
$$= (x+2)^2 - 9$$
$$\boxed{\therefore p = 2,\ q = -9}$$

(b) part (a) hints at the use of integration by completing the square for part (b)

$$\int \frac{1}{\sqrt{(x+2)^2 - 9}}\,dx$$

looking for a general form for above in the formula book

↳ see $\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \text{arcosh}\left(\frac{x}{a}\right)$

∴ substituting what's given to us into the formula

$x \to x + 2$

and $a^2 = 9$

$a = 3$

$$\int \frac{1}{\sqrt{(x+2)^2 - 3^2}}\,dx = \boxed{\text{arcosh}\left(\frac{x+2}{3}\right)}$$

or if want to evaluate using formula for arcosh$x$

$$= \boxed{\ln\left(x + 2 + \sqrt{(x+2)^2 - 9}\right)}$$

Question 2 continued

(c) remembering formula for mean value of a function

mean value $= \frac{1}{b-a}\int_a^b f(x)\,dx$

subbing in 'a' = 3, 'b' = 13

$\frac{1}{13-3}\int_3^{13} \frac{1}{\sqrt{x^2+4x-5}}\,dx$

$= \frac{1}{10}\int_3^{13} \frac{1}{\sqrt{x^2+4x-5}}\,dx$

using part (b) evaluated in the limits

$\frac{1}{10}\left\{\left[\operatorname{arcosh}\left(\frac{13+2}{3}\right) - \operatorname{arcosh}\left(\frac{3+2}{3}\right)\right]\right\}$

$\frac{1}{10}\left\{\operatorname{arcosh}(5) - \operatorname{arcosh}(5/3)\right\}$

using arcosh x formula (formula book)

$\frac{1}{10}\left\{\ln(15+\sqrt{216}) - \ln(5+\sqrt{16})\right\}$

∴ using quotient log law:

$\frac{1}{10}\ln\left(\frac{15+\sqrt{216}}{5+\sqrt{16}}\right)$

$= \frac{1}{10}\ln\left(\frac{15+6\sqrt{6}}{9}\right)$

÷3

$= \boxed{\frac{1}{10}\ln\left(\frac{5+2\sqrt{6}}{3}\right)}$

(Total for Question 2 is 6 marks)

3. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

$$z_1 = -4 + 4\text{i}$$

(a) Express z_1 in the form $r(\cos\theta + \text{i}\sin\theta)$, where $r \in \mathbb{R}$, $r > 0$ and $0 \leqslant \theta < 2\pi$ **(2)**

$$z_2 = 3\left(\cos\frac{17\pi}{12} + \text{i}\sin\frac{17\pi}{12}\right)$$

(b) Determine in the form $a + \text{i}b$, where a and b are exact real numbers,

(i) $\dfrac{z_1}{z_2}$ **(2)**

(ii) $(z_2)^4$ **(2)**

(c) Show on a single Argand diagram

(i) the complex numbers z_1, z_2 and $\dfrac{z_1}{z_2}$

(ii) the region defined by $\{z \in \mathbb{C} : |z - z_1| < |z - z_2|\}$ **(4)**

(a) currently z_1 given in 'a+bi' form - to get in mod-arg form means need to find $r_{z_1} = |z_1|$ and $\arg(z_1) = \theta$

mod $|z_1| = \sqrt{(-4)^2 + (4)^2}$

$= \sqrt{32} = \sqrt{16\times2}$

$= 4\sqrt{2}$

and

$\arg(z_1)$ - following general formula for arg

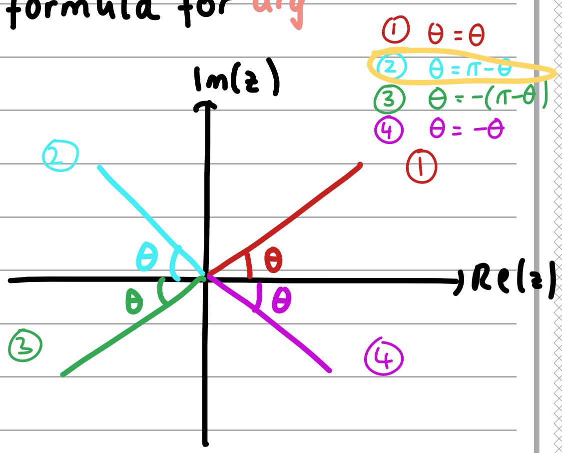
$\arg(z_1) = \tan^{-1}\left(\frac{y}{x}\right)$

$= \tan^{-1}\left(\frac{4}{4}\right)$

$= \tan^{-1}(1)$

$= \pi/4$

and find the right corresponding angle to the point $(-4,4)$ on the Argand diagram (in quadrant 2) $\therefore \arg(z_1) = \pi - \pi/4 = 3\pi/4$



DO NOT WRITE IN THIS AREA

Question 3 continued

$\therefore\ z_1 = 4\sqrt{2}\left(\cos(3\pi/4) + i\sin\left(\frac{3\pi}{4}\right)\right)$

(or if didn't simplify the surd:

$= \boxed{\sqrt{32}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)}$

(b) METHOD 1: dividing in mod-arg and converting ANS to a+bi

remembering the formula for dividing complex numbers expressed in mod-arg form: $\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right)$ [divide mod subtract args]

using z_1 from (a) and given z_2

$$\frac{z_1}{z_2} = \frac{4\sqrt{2}}{3}\left(\cos\left(\frac{3\pi}{4} - \frac{17\pi}{12}\right) + i\sin\left(\frac{3\pi}{4} - \frac{17\pi}{12}\right)\right)$$

$$= \frac{4\sqrt{2}}{3}\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$$

can evaluate this on CALC-CLASSWIZ using 2.complex and typing above in to get a+bi or evaluate $\cos\left(-\frac{2\pi}{3}\right)$ and $\sin\left(-\frac{2\pi}{3}\right)$ separately, multiplying by $\frac{4\sqrt{2}}{3}$

$$\cos\left(-\frac{2\pi}{3}\right) = -\frac{1}{2}$$

$$\sin\left(-\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$= \frac{\cancel{4}^{2}\sqrt{2}}{3}\left(-\frac{1}{\cancel{2}} + i\left(-\frac{\sqrt{3}}{\cancel{2}}\right)\right)$$

expand

$$= -\frac{2\sqrt{2}}{3} + i\left(\frac{2\sqrt{2}(-\sqrt{3})}{3}\right)$$

$$= -\frac{2\sqrt{2}}{3} + i\left(-\frac{2\sqrt{6}}{3}\right)$$

$$= \boxed{-\frac{2\sqrt{2}}{3} - i\frac{2\sqrt{6}}{3}}$$

METHOD 2: can divide in exponential form to then convert answer to mod-arg and finally to a+bi form

consider $z_1 = 4\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$ in exponential form: $re^{i\theta}$

$$\therefore z_1 = 4\sqrt{2}e^{i3\pi/4}$$

consider $z_2 = 3\left(\cos\left(\frac{17\pi}{12}\right) + i\sin\left(\frac{17\pi}{12}\right)\right)$ in exponential form: $re^{i\theta}$

$$\therefore z_2 = 3e^{\frac{17\pi}{12}}$$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

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∴ using formula for dividing exponentials: $\frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$

$$\frac{z_1}{z_2} = \frac{4\sqrt{2}}{3}e^{i\left(\frac{3\pi}{4}-\frac{17\pi}{12}\right)}$$

$$= \frac{4\sqrt{2}}{3}e^{i(-2\pi/3)}$$

↳ need in a+bi form ∴ first into mod-arg: $r(\cos\theta + i\sin\theta)$

$$= \frac{4\sqrt{2}}{3}\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$$

now can evaluate this either using 2.complex numbers on CALC CLASSWIZ by typing the above OR evaluate $\cos(-2\pi/3)$ and $\sin(-2\pi/3)$ separately - multiply both by $\frac{4\sqrt{2}}{3}$

$$\cos\left(-\frac{2\pi}{3}\right) = -\frac{1}{2}$$

$$\sin\left(-\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

subbing into above

$$= \frac{\cancel{4}^{2}\sqrt{2}}{3}\left(-\frac{1}{\cancel{2}} + i\left(-\frac{\sqrt{3}}{\cancel{2}}\right)\right)$$

expand

$$= \frac{-2\sqrt{2}}{3} - i\frac{(2\sqrt{3}\times\sqrt{2})}{3}$$

$$= \boxed{\frac{-2\sqrt{2}}{3} - \frac{2\sqrt{6}}{3}i}$$

METHOD 3: 'rationalising' the quotient of the two complex numbers in a+bi form i.e multiplying complex number in the denominator by its complex conjugate

want to divide both in a+bi form ∴ need to convert z_2 (given in mod-arg form) into a+bi form

either evaluate using 2.complex numbers on CALC-CLASSWIZ by typing above in OR evaluate $\cos\left(\frac{17\pi}{12}\right)$ and $\sin\left(\frac{17\pi}{12}\right)$ separately multiplying by 3

$$\cos\left(\frac{17\pi}{12}\right) = \frac{\sqrt{2}-\sqrt{6}}{4}$$

$$\sin\left(\frac{17\pi}{12}\right) = \frac{\sqrt{2}-\sqrt{6}}{4}$$

subbing into mod-arg

$$z_2 = 3\left(\frac{\sqrt{2}-\sqrt{6}}{4} + i\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)\right)$$

using $\cos\theta$ even function : $\cos(\theta) = \cos(-\theta)$

and $\sin\theta$ odd function : $\sin(\theta) = -\sin(\theta)$

$$3\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right) - i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)$$

$$\therefore \frac{z_1}{z_2} = \frac{-4+4i}{3\left(\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right) - i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\right)} \times \frac{\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right) + i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)}{\left(\frac{\sqrt{2}-\sqrt{6}}{4} + i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\right)}$$

numerator:

$$(-4+4i)\left(\frac{\sqrt{2}-\sqrt{6}}{4} + i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\right)$$

expand brackets

$$-\cancel{4}\left(\frac{\sqrt{2}-\sqrt{6}}{\cancel{4}}\right) + (-\cancel{4})i\left(\frac{\sqrt{2}+\sqrt{6}}{\cancel{4}}\right) + \cancel{4}i\left(\frac{\sqrt{2}-\sqrt{6}}{\cancel{4}}\right) + \cancel{4}i^2\left(\frac{\sqrt{2}+\sqrt{6}}{\cancel{4}}\right)$$

using $i = \sqrt{-1} \Rightarrow i^2 = -1$

$$= -(\sqrt{2}-\sqrt{6}) - i(\sqrt{2}+\sqrt{6}) + i(\sqrt{2}-\sqrt{6}) - (\sqrt{2}+\sqrt{6})$$

expand

$$= -\sqrt{2} + \cancel{\sqrt{6}} - \cancel{i\sqrt{2}} - i\sqrt{6} + \cancel{i\sqrt{2}} - i\sqrt{6} - \sqrt{2} - \cancel{\sqrt{6}}$$

$$= -2\sqrt{2} - 2\sqrt{6}\,i$$

denominator:

$$3 \times \left(\frac{\sqrt{2}-\sqrt{6}}{4} - i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\right)\left(\frac{\sqrt{2}-\sqrt{6}}{4} + i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\right)$$

expand

$$3 \times \left[\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right) + \cancel{i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)} - \cancel{i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)} - i^2\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\right]$$

$$= 3\left[\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right) + \left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\right]$$

evaluate on calc

$$= 3\left[\frac{2-\cancel{\sqrt{3}}}{\cancel{4}} + \frac{2+\cancel{\sqrt{3}}}{\cancel{4}}\right]$$

$$= 3\left[\frac{1}{2} + \frac{1}{2}\right]$$

$$= 3\,[1]$$

$$= 3$$

$$\boxed{\therefore \frac{z_1}{z_2} = \frac{-2\sqrt{2}}{3} - \frac{2\sqrt{6}}{3}\,i}$$

(ii) $(z_2)^4 = 3^4\left(\cos\left(\frac{17\pi}{12}\right) + i\sin\left(\frac{17\pi}{12}\right)\right)^4$

METHOD 1: applying De Moivre's theorem to mod-arg form of z_2

given as $z_2 = 3\left(\cos\left(\frac{17\pi}{12}\right) + i\sin\left(\frac{17\pi}{12}\right)\right)$

applying De Moivre's theorem: $z^n = r^n(\cos n\theta + i\sin n\theta)$

$\Rightarrow 3^4\left(\cos\left(\cancel{4}^{1} \times \frac{17\pi}{\cancel{12}_{3}}\right) + i\sin\left(\cancel{4}^{1} \times \frac{17\pi}{\cancel{12}_{3}}\right)\right)$

$\Rightarrow 81\left(\cos\left(\frac{17\pi}{3}\right) + i\sin\left(\frac{17\pi}{3}\right)\right)$

evaluate on calculator

$\Rightarrow 81\left(\frac{1}{2} - i\left(\frac{\sqrt{3}}{2}\right)\right)$

$\Rightarrow (z_2)^4 = \boxed{\frac{81}{2} - \frac{81\sqrt{3}}{2}i}$

METHOD 2: apply DMT to z_2 in exponential form: $re^{i\theta}$

from part (i)

$z_2 = 3e^{i(17\pi/12)}$

$\therefore$ applying DMT: $z^n = r^n e^{i(n\theta)}$

$= (z_2)^4 = 3^4 e^{i(\cancel{4}(17\pi/\cancel{12}_{3}))}$

$= 81\, e^{i(17\pi/3)}$

but need this in $a+bi$ form,

so first into mod-arg

$= 81\left(\cos\left(\frac{17\pi}{3}\right) + i\sin\left(\frac{17\pi}{3}\right)\right)$

either evaluate straight away on calc - CLASSWIZ

or evaluate $\cos\left(\frac{17\pi}{3}\right)$ and $\sin\left(\frac{17\pi}{3}\right)$ separately, multiplying by 81

$\cos\left(\frac{17\pi}{3}\right) = \frac{1}{2}$

$\sin\left(\frac{17\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$= 81\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

expand

$= \boxed{\frac{81}{2} - \frac{81\sqrt{3}}{2}i}$

METHOD 3: binomial expansion in $a+bi$ form

use $a+bi$ form of z_2 from part (i) method 3

$z_2 = 3\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right) - i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)$

and attempting to take all to the power of 4

$(z_2)^4 = 3^4\left(\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right) - i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\right)^4$

$$(z_4)^4 = 81\left(\left(\frac{\sqrt{2}-\sqrt{6}}{4} - i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\right)\right)^4$$

using Binomial expansion

Pascal's triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$$= 81\left[1\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)^4\left(-i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\right)^0 + 4\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)^3\left(-i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\right)^1 + 6\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)^2\left(-i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\right)^2\right.$$

$$\left. + 4\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)^1\left(-i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\right)^3 + 1\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)^0\left(-i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\right)^4\right]$$

(1) (2) (3) (4) (5) (6) (7) (8) (9) (10)

...expand separately:

(1) $\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)^4 = \frac{(\sqrt{2}-\sqrt{6})^4}{4^4} = \frac{(\sqrt{2}-\sqrt{6})^4}{256}$

$(\sqrt{2}-\sqrt{6})^4 = (\sqrt{2}-\sqrt{6})^2(\sqrt{2}-\sqrt{6})^2$

$(\sqrt{2}-\sqrt{6})^2 = 2 - 2\sqrt{12} + 6$

$= 8 - 2\sqrt{3\times 4}$

$= 8 - 4\sqrt{3}$

$\Rightarrow (\sqrt{2}-\sqrt{6})^4 = (8-4\sqrt{3})(8-4\sqrt{3})$

expand

$= 64 - 64\sqrt{3} + 16(3)$

$= 112 - 64\sqrt{3}$

$\therefore \left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)^4 = \frac{112-64\sqrt{3}}{256}$ $\div 16$ $\div 16$

$= \frac{7-4\sqrt{3}}{16}$

(2) $\left(-i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\right)^0 = 1$

(3) $\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)^3 = \frac{(\sqrt{2}-\sqrt{6})^3}{4^3} = \frac{(\sqrt{2}-\sqrt{6})^3}{64}$

BINOMIAL EXPANSION

1
1 1
1 2 1
1 3 3 1

$(\sqrt{2}-\sqrt{6})^3 = 1(\sqrt{2})^3 + 3(\sqrt{2})^2(-\sqrt{6}) + 3(\sqrt{2})(-\sqrt{6})^2 + 1(-\sqrt{6})^3$

$= 2\sqrt{2} + 3(2)(-\sqrt{6}) + 3\sqrt{2}(6) - 6\sqrt{6}$

collect like terms

$= 20\sqrt{2} - 12\sqrt{6}$

$\therefore \left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)^3 = \frac{20\sqrt{12}-12\sqrt{6}}{64}$ $\div 64$ $\div 64$ $= \frac{5\sqrt{2}-3\sqrt{6}}{16}$

(4) $\left(-i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\right)^1 = i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)$

(5) $\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)^2 = \frac{(\sqrt{2}-\sqrt{6})^2}{4^2} = \frac{(\sqrt{2}-\sqrt{6})^2}{16}$

$(\sqrt{2}-\sqrt{6})^2 = 8-4\sqrt{3}$ (prev.)

$$\therefore \left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)^2 = \frac{8-4\sqrt{3}}{16}\,\frac{\div 4}{\div 4} = \frac{2-\sqrt{3}}{4}$$

⑥ $$\left(-i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\right)^2 = (-i)^2\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)^2 = -\frac{(\sqrt{2}+\sqrt{6})^2}{16}$$

$$\begin{aligned}(\sqrt{2}+\sqrt{6})^2 &= (\sqrt{2})^2+2\sqrt{12}+(\sqrt{6})^2\\ &= 2+6+2\sqrt{4\times 3}\\ &= 8+4\sqrt{3}\end{aligned}$$

$$\therefore \left(-i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\right)^2 = -\frac{8+4\sqrt{3}}{16}\ \div 4$$

$$= -\left(\frac{2+\sqrt{3}}{4}\right)$$

⑦ $$\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)^1 = \left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)$$

⑧ $$\left(-i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\right)^3 = (-i)^3\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)^3 = -i\,\frac{(-\sqrt{2}+\sqrt{6})^3}{64}$$

$(\sqrt{2}+\sqrt{6})^3$ - BINOMIAL EXPANSION

$$\begin{array}{c}1\\1\quad 1\\1\quad 2\quad 1\\1\quad 3\quad 3\quad 1\end{array}\qquad \begin{aligned}&= 1(\sqrt{2})^3+3(\sqrt{2})^2(\sqrt{6})+3(\sqrt{2})(\sqrt{6})^2+1(\sqrt{6})^3\\ &= 2\sqrt{2}+6\sqrt{6}+18\sqrt{2}+6\sqrt{6}\\ &= 20\sqrt{12}+12\sqrt{6}\end{aligned}$$

$$\therefore \left(-i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\right)^3 = -i\left(-\frac{20\sqrt{2}+12\sqrt{6}}{64}\right)\ \div 4\ \div 4$$

$$= -i\left(-\left(\frac{5\sqrt{2}+3\sqrt{6}}{16}\right)\right)$$

⑨ $$\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)^0 = 1$$

⑩ $$\left(-i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\right)^4 = (-i)^4\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)^4 = \frac{(\sqrt{2}+\sqrt{6})^4}{256}$$

$$\begin{aligned}(\sqrt{2}+\sqrt{6})^4 &= \left((\sqrt{2}+\sqrt{6})^2\right)^2\\ &= \left((8+4\sqrt{3})\right)^2\\ &= (8+4\sqrt{3})(8+4\sqrt{3})\\ &= 64+64\sqrt{3}+48\\ &= 112+64\sqrt{3}\end{aligned}$$

$$\therefore \left(-i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)^4\right) = \frac{112+64\sqrt{3}}{256}\ \div 16 = \frac{7+4\sqrt{3}}{16}$$

subbing all into Binomial expansion:

$$81\left[\left(\frac{7-4\sqrt{3}}{16}\right)+4\left(\frac{5\sqrt{2}-3\sqrt{6}}{16}\right)\left(-i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\right)^{①}+6\left(\frac{2-\sqrt{3}}{4}\right)\left(-\frac{(2+\sqrt{3})}{4}\right)^{②}+4\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)\left(-i\left(-\left(\frac{5\sqrt{2}+3\sqrt{6}}{16}\right)\right)\right)^{③}+\frac{7+4\sqrt{3}}{16}\right]$$

now expanding from above:

① : $4\left(\frac{5\sqrt{2}-3\sqrt{6}}{16}\right)\left(-i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\right)$

$= -4i\left(\frac{5\sqrt{2}-3\sqrt{6}}{16}\times\frac{\sqrt{2}+\sqrt{6}}{4}\right)$

$= -\left(\frac{-2+\sqrt{3}}{4}\right)i$

② $6\left(\left(\frac{2-\sqrt{3}}{4}\right)\left(-\frac{2+\sqrt{3}}{4}\right)\right)$

$= -3/8$

③ $4\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)\left(-i\left(-\frac{5\sqrt{2}+3\sqrt{6}}{16}\right)\right)$

$= -\frac{2+\sqrt{3}}{4}i$

subbing in

$$81\left[\frac{7-4\sqrt{3}}{16}-\left(\frac{-2+\sqrt{3}}{4}\right)i-3/8-\left(\frac{2+\sqrt{3}}{4}\right)i+\left(\frac{7+4\sqrt{3}}{16}\right)\right]$$

...real:

$\frac{7-4\sqrt{3}}{16}+\frac{7+4\sqrt{3}}{16}-\frac{3(2)}{16}$

$=\frac{14-6}{16}=\frac{8}{16}=1/2$

...imaginary:

$-\left(\frac{-2+\sqrt{3}}{4}i\right)-\left(\frac{2+\sqrt{3}}{4}\right)i$

$=-\sqrt{3}/2\, i$

$\therefore 81\left(1/2-\sqrt{3}/2\, i\right)$

$=\boxed{\frac{81}{2}-\frac{81\sqrt{3}}{2}i}$

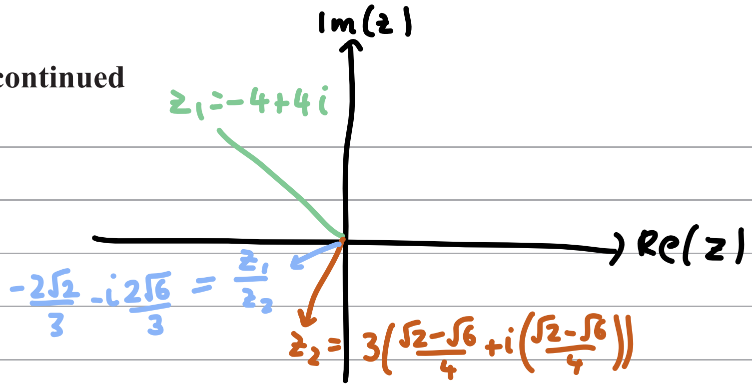
(c) remembering how on an Argand diagram complex numbers of form $a+bi$ are mapped with Cartesian coordinates (a,b)

$\therefore z_1=-4+4i$

$z_2=3\left(\frac{\sqrt{2}-\sqrt{6}}{4}+i\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)\right)$

and $\frac{z_1}{z_2}=-\frac{2\sqrt{2}}{3}-i\frac{2\sqrt{6}}{3}$ → evaluate surds as decimals (on calc) to make easier to plot!

Question 3 continued

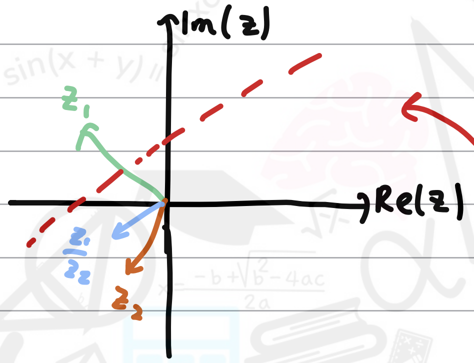


(ii) noticing how region given in form:

$|z - z_1| = |z - z_2|$

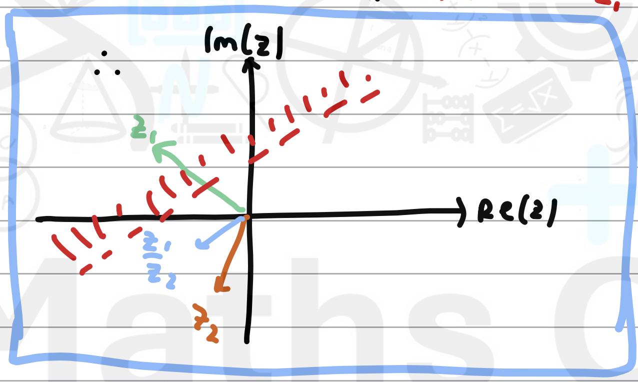
which represents the perpendicular bisector between the two points

$z_1 = (-4, 4)$ and $z_2 = \left(\frac{3\sqrt{2}-\sqrt{6}}{4}, \frac{3\sqrt{2}-\sqrt{6}}{4}\right)$



· so we know the shape - need to see if dashed or solid - '<' ∴ dashed

· now which side to SHADE - one closest to z_1 (<)



(Total for Question 3 is 10 marks)

4. Prove by induction that for $n \in \mathbb{N}$

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & -2n \\ 0 & 1 \end{pmatrix}$$

(5)

(a) Proving by mathematical induction involves proving a conjecture is true for all $n \in \mathbb{N}$

step 1: base case

prove true for $n=1$

LHS: $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^1 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$

RHS: $\begin{pmatrix} 1 & -2(1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$

LHS = RHS $\therefore$ true for $n=1$

Step 2: assumption case

assume true for $n=k$

$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & -2k \\ 0 & 1 \end{pmatrix}$

Step 3: induction step

prove true for $n=k+1$

$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^k \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & -2k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$

AIM: $\begin{pmatrix} 1 & -2(k+1) \\ 0 & 1 \end{pmatrix}$

using manual matrix multiplication due to unknown 'k'

↳ "rows into columns" ⇒ let product matrix be $\begin{pmatrix} A_{(1,1)} & A_{(1,2)} \\ A_{(2,1)} & A_{(2,2)} \end{pmatrix}$

... for $A_{(1,1)}$:

$(1)(1) - 2k(0)$

$= 1$ (✓ aim)

... for $A_{(1,2)}$:

$= 1(-2) + 1(-2k)$

$= -2 - 2k$

or $-2k - 2$ want to make look like AIM $\therefore$ factorise -2 out

$= -2(k+1)$ (✓ aim)

P 7 2 7 9 4 A 0 1 0 2 8

Question 4 continued

... for $A_{(2,1)}$:

$(1)(0) + (0)(1)$

$= 0$ (✓ aim)

...for $A_{(2,2)}$:

$(0)(-2) + 1(1) = 1$ (✓ aim)

$\therefore$ true for $n = k+1$

step 4: conclusion

since true for $n=1$, if true for $n=k$ and true for $n=k+1$, then true for all $n \in \mathbb{N}$

(Total for Question 4 is 5 marks)

5. The line l_1 has equation $\dfrac{x+5}{1} = \dfrac{y+4}{-3} = \dfrac{z-3}{5}$

The plane Π_1 has equation $2x + 3y - 2z = 6$

(a) Find the point of intersection of l_1 and Π_1

(2)

The line l_2 is the reflection of the line l_1 in the plane Π_1

(b) Show that a vector equation for the line l_2 is

$$\mathbf{r} = \begin{pmatrix} -7 \\ 2 \\ -7 \end{pmatrix} + \mu \begin{pmatrix} 10 \\ 6 \\ 2 \end{pmatrix}$$

where μ is a scalar parameter.

(5)

The plane Π_2 contains the line l_1 and the line l_2

(c) Determine a vector equation for the line of intersection of Π_1 and Π_2

(2)

The plane Π_3 has equation $\mathbf{r}.\begin{pmatrix} 1 \\ 1 \\ a \end{pmatrix} = b$ where a and b are constants.

Given that the planes Π_1, Π_2 and Π_3 form a sheaf,

(d) determine the value of a and the value of b.

(3)

(a) notice how both the equation of the line (l_1) and equation of the plane (π) given in Cartesian form

METHOD 1: subbing l_1 parametric form into π_1 cartesian + solve for λ

if we convert Cartesian equation of line to vector parametric form $r = a + \lambda b$ - can get general expression for the 'x', 'y', 'z' coordinates that can then sub into the cartesian equation of π_1 to find parameter which will take us to a specific point of intersection

taking Cartesian general equation

$$\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3} = \lambda \quad \text{where } r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \begin{matrix} = a_1 + \lambda b_1 \\ = a_2 + \lambda b_2 \\ = a_3 + \lambda b_3 \end{matrix}$$

and subbing in information from question

$$\frac{x+5}{1} = \frac{y+4}{-3} = \frac{z-3}{5} = \lambda$$

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Question 5 continued

...position vectors: ...direction vectors:

$-a_1 = 5$ $b_1 = 1$

$\Rightarrow a_1 = -5$ $b_2 = -3$

$-a_2 = 4$ $b_3 = 5$

$\Rightarrow a_2 = -4$

$-a_3 = -3$

$\Rightarrow a_3 = 3$

$$\therefore l_1 = \begin{pmatrix} -5 \\ -4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$$

... general coordinate:

$$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5+\lambda \\ -4-3\lambda \\ 3+5\lambda \end{pmatrix}$$

...form linear equations:

$x = -5+\lambda$

$y = -4-3\lambda$

$z = 3+5\lambda$

and subbing these into Cartesian equation of Π_1, to get λ

$\Pi_1 = 2(-5+\lambda) + 3(-4-3\lambda) - 2(3+5\lambda) = 6$

expand out

$= -10 + 2\lambda - 12 - 9\lambda - 6 - 10\lambda = 6$

taking λs to LHS, integers to RHS

$= \lambda(2-9-10) = 10+12+6+6$

$= -17\lambda = 34$

$\div -17$ $\div -17$

$\Rightarrow \lambda = -2$

subbing parameter into general coordinate

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5+(-2) \\ -4-3(-2) \\ 3+5(-2) \end{pmatrix}$$

$$= \begin{pmatrix} -7 \\ 2 \\ -7 \end{pmatrix}$$ $\boxed{\therefore \text{p.o.i} = (-7, 2, -7)}$

METHOD 2: (preferred considering REFLECTION in part (b))

converting plane into $r.n = p$ and subbing general equation of l_1 into 'r' to get λ

Π_1 in Cartesian equation: $n_1x + n_2y + n_3z = p$ - but need in scalar form $r.n = p$ - subbing in Π_1 in question, deduce that:

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$$

and $p = 6$

Π_1 in scalar form: $r.\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} = 6$

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now subbing in general vector parametric form of ℓ_1 (Cartesian → vector parametric; negate position vectors (numerator) and keep denominator as direction vector

$$\ell_1 = \begin{pmatrix} -5 \\ -4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$$

∴ general coordinate

$$\ell_1 = r = \begin{pmatrix} -5+\lambda \\ -4-3\lambda \\ 3+5\lambda \end{pmatrix}$$

sub into π_1 in scalar form :

$$\begin{pmatrix} -5+\lambda \\ -4-3\lambda \\ 3+5\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} = 6$$

and dot product (sum of products of all corresponding elements)

$$2(-5+\lambda) + 3(-4-3\lambda) + (-2)(3+5\lambda) = 6$$

expand

$$-10+2\lambda-12-9\lambda-6-10\lambda = 6$$

collect like terms

$$\lambda(2-9-10) = 10+12+6+6$$

$$\Rightarrow -17\lambda = 34$$

$\div -17$ $\quad$ $\div -17$

$\Rightarrow \lambda = -2$ → subbing 'λ' into general vector parametric of ℓ_1

$$\ell_1 = r = \begin{pmatrix} -5+(-2) \\ -4-3(-2) \\ 3+5(-2) \end{pmatrix} = \begin{pmatrix} -7 \\ 2 \\ -7 \end{pmatrix}$$

$\boxed{\therefore \text{p.o.i} = (-7, 2, -7)}$

METHOD 3: substitution ℓ_1 equation in terms of 'x' which can then sub into π_1 to solve simultaneously

let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = $ p.o.i

using Cartesian equation of ℓ_1 to write 'y' and 'z' in terms of 'x' which sub into π_1 cartesian to get 'x'

$$x = x \quad -①$$

$$\frac{x+5}{1} = \frac{y+4}{-3}$$

cross multiply and rearrange to make 'y' the subject

$$-3(x+5) = y+4$$

$$\Rightarrow y = -3x-19 \quad -②$$

$$\frac{x+5}{1} = \frac{z-3}{5}$$

cross multiply and rearrange to make 'z' subject

$5(x+5) = z-3$

$\Rightarrow \underline{z = 5x+28}$ — ③

subbing these equations for 'x', 'y' and 'z' into plane π_1

$2(x)+3(-3x-19)-2(5x+28)=6$

expand

$2x-9x-57-10x-56=6$

take x's to RHS

$-57-56-6=x(10+9)$

$\Rightarrow 19x=-199$

$\div 19 \quad \div 19$

$x=-7$

for 'y' substitute x into ②

$y=-3(-7)-19$

$=2$

for 'z' substitute in $x=-7$ into ③

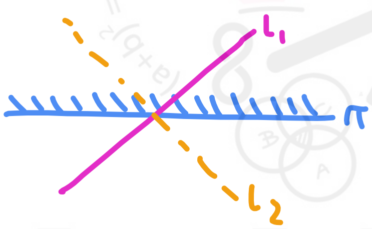
$z=5(-7)+28$

$=-35+28$

$=-7$

$\boxed{\text{p.o.i} = (-7, 2, -7)}$

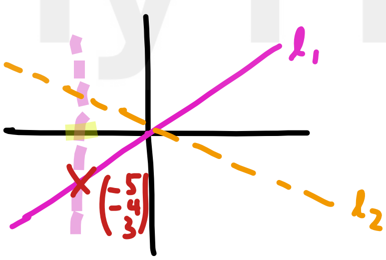
(b) question is asking us to reflect line l_1 with vector parametric equation with $r=\begin{pmatrix}-5\\-4\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\-3\\5\end{pmatrix}$ in the plane π_1 with cartesian equation $2x+3y-2z=6$ (scalar form $r\cdot\begin{pmatrix}2\\3\\-2\end{pmatrix}=6$ (from METHOD 2 (a))



key here always is to find TWO POINTS that would lie on l_2 and then form a vector parametric equation from these two points

↳ one of the points: the p.o.i between l_1 and π ($\begin{pmatrix}-7\\2\\-7\end{pmatrix}$) - from (a)

second would be the reflection of a single known point on l_2 - here it's position vector so $\begin{pmatrix}-5\\-4\\3\end{pmatrix}$ - through the plane π_1 to get image point on l_2



to do this can do:-

WAY 1: using vector equation of line perpendicular to plane

· finding vector parametric equation of line perpendicular to plane through the point $\begin{pmatrix}-5\\-4\\3\end{pmatrix}$ in form $a+tb$ ← direction; a ↓ position

↳ know that direction vector perpendicular to direction vector of plane ∴ by definition would be equal to the normal to plane → $\begin{pmatrix}2\\3\\-2\end{pmatrix}$

and position vector $=\begin{pmatrix}-5\\-4\\3\end{pmatrix}$

∴ equation perpendicular: $r=\begin{pmatrix}-5\\-4\\3\end{pmatrix}+t\begin{pmatrix}2\\3\\-2\end{pmatrix}$

$r=\begin{pmatrix}-5+2t\\-4+3t\\3-2t\end{pmatrix}$

· next need 't' where equation meets plane in scalar form: $r \cdot n = p$

subbing 'r'

$$\begin{pmatrix} -5+2t \\ -4+3t \\ 3-2t \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} = 6$$

dot product

$$2(-5+2t)+3(-4+3t)-2(3-2t)=6$$

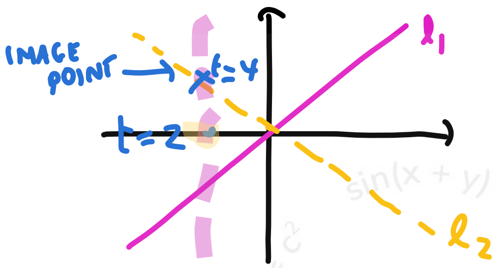
$$\Rightarrow -10+4t-12+9t-6+4t=6$$

collect like terms

$$\Rightarrow t(4+9+4)=6+10+12+6$$

$$\Rightarrow 17t = 34 \quad \div 17$$

$$\Rightarrow t = 2$$



$t=2$ is where the equation meets plane ∴ need a further 2 units to get on l_2

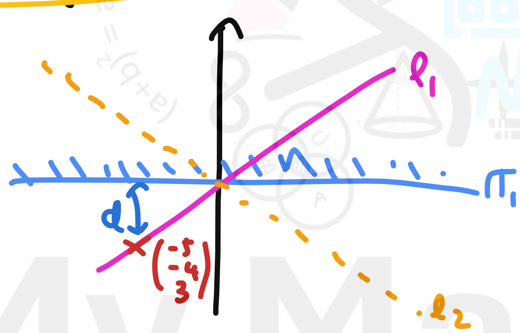
∴ image point must be where $t=4$

∴ subbing this parameter into the perpendicular general formula

$$r = \begin{pmatrix} -5+2(4) \\ -4+3(4) \\ 3-2(4) \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}$$

∴ second point is $\begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}$

WAY 2: using twice the perpendicular distance from the point $\begin{pmatrix} -5 \\ -4 \\ 3 \end{pmatrix}$ to plane π and doubling it to get image point on l_2



using equation in the formula book:

perp. distance of (α, β, γ) from $n_1x+n_2y+n_3z+d=0$ is $\dfrac{|n_1\alpha+n_2\beta+n_3\gamma|}{\sqrt{(n_1)^2+(n_2)^2+(n_3)^2}}$

where $(\alpha, \beta, \gamma) = (-5, -4, 3)$ and $\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$

$$d = \frac{|2(-5)+3(-4)-2(3)-6|}{\sqrt{(2)^2+(3)^2+(-2)^2}} = \frac{|-34|}{\sqrt{4+9+4}} = \frac{34}{\sqrt{17}} \quad \begin{matrix} \times\sqrt{17} \\ \times\sqrt{17} \end{matrix}$$

rationalise

$$d = \frac{\cancel{34}^{\,2}\sqrt{17}}{\cancel{17}}$$

$$\Rightarrow d = 2\sqrt{17}$$

hence we know that the second general point on l_2 is $2d = 4\sqrt{17}$ units away from $\begin{pmatrix} -5 \\ -4 \\ 3 \end{pmatrix}$

∴ using normal of a plane:

$$\left|\begin{pmatrix} 2k \\ 3k \\ -2k \end{pmatrix}\right| = 4\sqrt{17}$$

$$\sqrt{(2k)^2+(3k)^2+(-2k)^2} = 4\sqrt{17}$$

expand inside root:

$$\sqrt{4k^2+9k^2+4k^2} = 4\sqrt{17}$$

$\Rightarrow \sqrt{17k^2} = 4\sqrt{17}$

square both sides

$\Rightarrow 17k^2 = 272$

$\div 17 \quad \div 17$

$\Rightarrow k^2 = 16$

$\Rightarrow k = \pm 4$

$\therefore k = 4$

using equation perpendicular through $(-5,-4,3)$ as position vector and direction vector normal to plane

$$r = \begin{pmatrix} -5+2k \\ -4+3k \\ 3-2k \end{pmatrix}$$

Sub in $k=4$

$$r = \begin{pmatrix} -5+2(4) \\ -4+3(4) \\ 3-2(4) \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}$$

$\Rightarrow$ image point is $\begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}$

FINALLY once we've got the two points on l_2 - can form vector equation: $r = a + \lambda b$

let $A = \begin{pmatrix} -7 \\ 2 \\ -7 \end{pmatrix}$ and $B = \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}$

could take any as 'a' but in 'show that' this is $\begin{pmatrix} -7 \\ 2 \\ -7 \end{pmatrix}$ and for direction vector $b = \vec{AB} = \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix} - \begin{pmatrix} -7 \\ 2 \\ -7 \end{pmatrix}$

$$= \begin{pmatrix} 10 \\ 6 \\ 2 \end{pmatrix}$$

$$\boxed{\therefore l_2 = \begin{pmatrix} -7 \\ 2 \\ -7 \end{pmatrix} + \mu \begin{pmatrix} 10 \\ 6 \\ 2 \end{pmatrix}}$$

(c) METHOD 1: using line of mirror points from (b) = QUICKEST + best method

to find vector equation $r = a + \lambda b$ of line of intersection - need to find two common points and vector through them

↳ see how $\begin{pmatrix} -7 \\ 2 \\ -7 \end{pmatrix}$ is common to both Π_1 and Π_2 (p.o.i from (a))

so that's one point

↳ for second point - can use the perpendicular line equation (that joining mirror lines) -

$$\begin{pmatrix} -5+2t \\ -4+6t \\ 3-2t \end{pmatrix}$$

but write $t=2$ from (b)

$$\begin{pmatrix} -5+2(2) \\ -4+6(2) \\ 3-2(2) \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

$\therefore$ call $A = \begin{pmatrix} -7 \\ 2 \\ -7 \end{pmatrix}$ and $B = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$

$$\vec{AB} = \begin{pmatrix} -1-(-7) \\ 2-2 \\ -1-(-7) \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 6 \end{pmatrix}$$

and take 'a' s any of A or B

$$\Rightarrow \boxed{r = \begin{pmatrix}-7\\2\\-7\end{pmatrix} + s\begin{pmatrix}6\\0\\6\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}-1\\2\\-1\end{pmatrix} + s\begin{pmatrix}1\\0\\1\end{pmatrix}}$$

METHOD 2: using Cartesian equations

generally for equation of line of intersection between 2 planes - need two common points and vector through them

- can find these two common points through using Cartesian equation of π_1 and π_2

$\pi_1 : 2x + 3y - 2z = 6$ (given)

π_2 : finding Cartesian equation $n_1x + n_2y + n_3z = d$ need to find NORMAL of π_2 and 'd' for $a \cdot n = p$

'contains ℓ_1 and ℓ_2 $\therefore$ normal must be perpendicular to 'b_1' and 'b_2''

let normal $= \begin{pmatrix}x\\y\\z\end{pmatrix}$

$\therefore$ using fact that for vector to be perpendicular to another

$a \cdot b = 0$

...for b_1:

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}1\\-3\\5\end{pmatrix} = x - 3y + 5z = 0 \quad -①$$

...for b_2:

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}10\\6\\2\end{pmatrix} = 10x + 6y + 2z = 0 \quad -②$$

let $z = 1$

$x - 3y = -5 \quad -③$

$10x + 6y = -2 \quad -④$

solve these simultaneously (CALC equ. solver or by elimination):

③×2 $\quad 2x - 6y = -10$

$+ \quad 10x + 6y = -2$

$12x = -12 \quad \div 12$

$\Rightarrow x = -1$

and sub into ③

$-1 - 3y = -5$

$\Rightarrow 3y = 4 \quad \div 3$

$y = 4/3$

$\therefore$ normal $= \begin{pmatrix}-1\\4/3\\1\end{pmatrix} \times 3 = \begin{pmatrix}-3\\4\\3\end{pmatrix}$

now that we've got normal, need 'p'

↳ from $a \cdot n = p$

where 'a' eg. $\begin{pmatrix}-7\\2\\-7\end{pmatrix}$ as lies on both π_1 and π_2

$$\begin{pmatrix}-7\\2\\-7\end{pmatrix} \cdot \begin{pmatrix}-3\\4\\3\end{pmatrix} = 21 + 8 - 21 = 8$$

now get Cartesian equation of π_2

$\pi_2 : -3x+4y+3z=8$

$\pi_1 : 2x+3y-2z=6$

$\pi_2 : -3x+4y+3z=8$

now need to find the two common points through π_1 and π_2

let $z=0$

$2x+3y=6$ - ①

$-3x+4y=8$ - ②

solve for 'x' or 'y' - equation solver or by elimination

① × 4 $\quad 8x+12y=24$

② × 3 $\quad -9x+12y=24$

$17x=0$

$\Rightarrow x=0$

sub into ①

$2(0)+3y=6$

$\Rightarrow 3y=6 \quad \div 3$

$\Rightarrow y=2$

$\therefore$ one common point : $\begin{pmatrix}0\\2\\0\end{pmatrix}$

let $z=1$

$2x+3y=8$ - ③

$-3x+4y=5$ - ④

Solve for 'x' or 'y' - equ. solver or by elimination

③×4 $\quad 8x+12y=32$

$-9x+12y=15$

$17x=17$

$\Rightarrow x=1$

sub into ③

$2(1)+3y=8$

$\Rightarrow 3y=6 \quad \div 3$

$\Rightarrow y=2$

$\therefore$ second point is $\begin{pmatrix}1\\2\\1\end{pmatrix}$

FINALLY call first point A and second B

$A=\begin{pmatrix}0\\2\\0\end{pmatrix} \quad \vec{AB}=\begin{pmatrix}1\\2\\1\end{pmatrix}-\begin{pmatrix}0\\2\\0\end{pmatrix}$

$B=\begin{pmatrix}1\\2\\1\end{pmatrix} \quad\quad = \begin{pmatrix}1\\0\\1\end{pmatrix}$

$\boxed{\therefore \text{line of intersection : } \begin{pmatrix}0\\2\\0\end{pmatrix}+s\begin{pmatrix}1\\0\\1\end{pmatrix}}$

METHOD 3 : using CROSS PRODUCT - first Cartesian equation for π_2 $n_1x+n_2y+n_3z=d$ where $\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix}$ perpendicular to b_1 and b_2 $\therefore$ cross product

$$\begin{vmatrix} i & j & k \\ 5 & 3 & 1 \\ 1 & -3 & 5 \end{vmatrix} = \begin{pmatrix} 18 \\ -24 \\ -18 \end{pmatrix} \div 6 = \begin{pmatrix} 3 \\ -4 \\ -3 \end{pmatrix}$$

$\therefore$ normal is $\begin{pmatrix} 3 \\ -4 \\ -3 \end{pmatrix}$

Next to find 'd': $a \cdot n = d$

a: p.o.i from (a)

$$\begin{pmatrix} -7 \\ 2 \\ -7 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \\ -3 \end{pmatrix} = -21 - 8 + 21 = -8$$

$\therefore$ Cartesian π_2

$3x - 4y - 3z = -8$

$\pi_1 : 2x + 3y - 2z = 6$

$\pi_2 : 3x - 4y - 3z = -8$

now for the equation of a line: $a + \lambda b$

for 'b': cross product

$$\begin{vmatrix} i & j & k \\ 3 & -4 & -3 \\ 2 & 3 & -2 \end{vmatrix} = \begin{bmatrix} 17 \\ 0 \\ 17 \end{bmatrix} \div 17 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

and for 'a' - need one common point between both π_1 and π_2:

let $z = 0$

$2x + 3y = 6 \quad -(1)$

$3x - 4y = -8 \quad -(2)$

equ. solver or by elimination:

$(1) \times 4 \quad 8x - 12y = 24$

$(2) \times 3 \quad + \; 9x - 12y = -24$

$17x = 0$

$\Rightarrow x = 0$

Sub into (1)

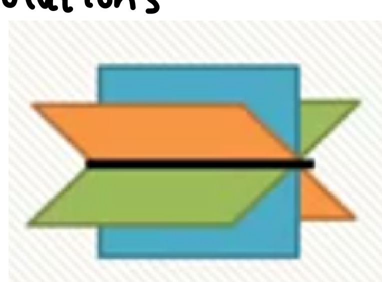
$\div 3 \quad 0 + 3y = 6 \quad \div 3$

$\Rightarrow y = 2$

$\therefore$ one common point $(0, 2, 0)$

$$\boxed{\Rightarrow \text{line} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$$

(d) planes form a Sheaf - from 3D MATRICES means have a line of intersection consisting of ∞ many points - system of equations consistent with ∞ many solutions



METHOD 1: using (c) i.e 'normal' facts and scalar form

by definition, line from (c) (intersection of π_1 and π_2 MUST LIE in the plane), so using fact that π_3 given in scalar form $r \cdot n = p$, $n = \begin{pmatrix} 1 \\ 1 \\ a \end{pmatrix}$ must be perpendicular to $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ from (c) i.e dot product $= 0$

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ a \end{pmatrix} = 1 + 0 + a = 0 \Rightarrow a + 1 = 0 \Rightarrow a = -1$$

now given 'a' can find 'b' using $a \cdot b = n$, where 'a' can be $\begin{pmatrix}-7\\2\\-7\end{pmatrix}$ as lies on all three planes

$$\begin{pmatrix}-7\\2\\-7\end{pmatrix} \cdot \begin{pmatrix}1\\1\\-1\end{pmatrix} = -7+2+7$$

$$\therefore b = 2$$

hence, $\boxed{a=-1, b=2}$

METHOD 2: using sim. equations then consistency

found using METHOD 2 part (c) that Cartesian equation of Π_2 is: $3x-4y-3z=-8$

this reminds us of 'solving linear equation' using 3D matrices

$\therefore$ if converted all three Πs into Cartesian - can use consistency and simultaneous equations to get 'a' and 'b'

$\Pi_1 = 2x+3y-2z=6$

Π_3 into Cartesian: $n_1x+n_2y+n_3z=d$ $\therefore$ using $\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix} = \begin{pmatrix}1\\1\\a\end{pmatrix}$ and $'d'=b$

$\Pi_3 = x+y+az=b$

because these are consistent, means have ∞ many solutions

$\therefore$ rearrange Π_3 to make 'x' the subject

$\Rightarrow x = b-y-az$

and sub into Π_1

$2(b-y-az)+3y-2z=6$

expand

$2b-2y-2az+3y-2z=6$

collect like terms

$y-(2a+2)z = 6-2b \quad -④$

sub into Π_2

$3(b-y-az)-4y-3z=-8$

expand

$3b-3y-3az-4y-3z=-8$

collect like terms

$-7y-z(3a+3)=-8-3b$

$\times -1 \qquad \times -1$

$7y+(3a+3)z=3b+8$

$\therefore$ consistency suggests multiply ④ by 7

$7y-7(2a+2)z=7(6-2b)$

$\Rightarrow 7y-7(2a+2)z=42-14b$

now can compare:

... z:

$3a+3=-14a-14$

$\Rightarrow 17a=-17$

$\div 17 \qquad \div 17$

$\Rightarrow a=-1$

... integers:

$42-14b=3b+8$

$\Rightarrow 17b=34$

$\div 17 \qquad \div 17$

$\Rightarrow b=2$

$\boxed{\therefore a=-1, b=2}$

Question 5 continued

METHOD 3: first finding normal to π_2 using vector product of the direction of l_1 and direction of l_2

normal to π_2 : $\begin{vmatrix} i & j & k \\ 1 & -3 & 5 \\ 10 & 6 & 2 \end{vmatrix} = \begin{pmatrix} -36 \\ 48 \\ 36 \end{pmatrix}$

now finding the determinant of the matrix of normal vectors i.e using

$n_1 = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$

$n_2 = \begin{pmatrix} -3 \\ 4 \\ 3 \end{pmatrix}$

$n_3 = \begin{pmatrix} 1 \\ 1 \\ a \end{pmatrix}$

and using fact that for π_1, π_2, π_3 to be a sheaf, $\det(A) = 0$ (infinite solutions)

$$\begin{vmatrix} 3 & -4 & 3 \\ 2 & 3 & -2 \\ 1 & 1 & a \end{vmatrix} = 0$$

$\Rightarrow 3(3a+2) + 4(2a+2) - 3(-1) = 0$

$\Rightarrow 9a + 6 + 8a + 8 + 3 = 0$

$\Rightarrow 17a = -17$

$a = -1$

then find 'b' using $r.n = b$

$\begin{pmatrix} -7 \\ 2 \\ -7 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = b$

$\Rightarrow -7 + 2 + 7 = b$

$\Rightarrow b = 2$

$\boxed{\therefore a = -1, b = 2}$

(Total for Question 5 is 12 marks)

6. Water is flowing into and out of a large tank.

Initially the tank contains 10 litres of water.

The rate of flow of the water is modelled so that

- there are V litres of water in the tank at time t minutes after the water begins to flow
- water enters the tank at a rate of $\left(3 - \dfrac{4}{1 + e^{0.8t}}\right)$ litres per minute
- water leaves the tank at a rate proportional to the volume of water remaining in the tank

Given that when $t = 0$ the volume of water in the tank is decreasing at a rate of 3 litres per minute, use the model to

(a) show that the volume of water in the tank at time t satisfies

$$\frac{dV}{dt} = 3 - \frac{4}{1 + e^{0.8t}} - 0.4V$$

(3)

(b) Determine $\dfrac{d}{dt}\left(\arctan e^{0.4t}\right)$

(2)

Hence, by solving the differential equation from part (a),

(c) determine an equation for the volume of water in the tank at time t.

Give your answer in simplest form as $V = f(t)$

(6)

After 10 minutes, the volume of water in the tank was 8 litres.

(d) Evaluate the model in light of this information.

(1)

(a) following the usual format for formulating 1 o.D.Es

rate of water in $= 3 - \frac{4}{1-e^{0.8t}}$

rate of water out $=$ "proportional to water remaining"

$= -kV$

$\therefore \frac{dV}{dt} =$ rate in $-$ rate out

$= 3 - \frac{4}{1+e^{0.8t}} - kV$

now plugging in initial conditions: when $t=0$, $V=10$, $\frac{dV}{dt} = -3$

$-3 = 3 - \frac{4}{1+e^0} - 10k$

evaluate e^0

$=1$ $-3 = 3 - \frac{4}{1+1} - 10k$

$=1$ $-3 = 3 - 2 - 10k$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Question 6 continued

$\therefore 10k = 6 - 2$

$10k = 4$

$\div 10 \qquad \div 10$

$k = \frac{\cancel{4}^{2}}{\cancel{10}_{5}}$

$= 2/5 = 0.4$

Sub back into initial $\frac{dV}{dt}$

$$\boxed{\therefore \frac{dV}{dt} = 3 - \frac{4}{1+e^{0.8t}} - 0.4V}$$

(b) METHOD 1: chain rule using formula book

given in formula book that $\frac{d}{dt}(\arctan x) = \frac{1}{1+x^2}$

$\therefore$ using $u = e^{0.4t}$ using rule: $\frac{d}{dt}(e^{kt}) = ke^{kt}$

$\frac{du}{dt} = 0.4e^{0.4t}$

$$\frac{d}{dt}(\arctan(e^{x})) = \frac{1}{1+(e^{0.4t})^2} \times 0.4e^{0.4t}$$

$$= \frac{1}{1+e^{0.8t}} \times 0.4e^{0.4t}$$

$$= \frac{0.4e^{0.4t}}{1+e^{0.8t}}$$ OR if 0.4 as a fraction $\boxed{\frac{2e^{0.4t}}{5(1+e^{0.8t})}}$

METHOD 2: implicit differentiation

let $y = \arctan(e^{0.4t})$

taking tan of both sides

$\tan y = \tan(\tan^{-1}(e^{0.4t}))$

$\Rightarrow \tan y = e^{0.4t}$

differentiate implicitly using $\frac{d}{dt}(\tan y) = \sec^2 y$ and $\frac{d}{dt}(e^{kt}) = ke^{kt}$

$\sec^2 y \frac{dy}{dt} = 0.4e^{0.4t}$

$\div \sec^2 y \qquad \div \sec^2 y$

$\Rightarrow \frac{dy}{dt} = \frac{0.4e^{0.4t}}{\sec^2 y}$

but need $\sec^2 y$ in terms of 't' $\therefore$ want to use $\tan y = e^{0.4t}$

Can use identity: $\sec^2 y = 1 + \tan^2 y$

$\sec^2 y = 1 + (e^{0.4t})^2$

$= 1 + e^{0.8t}$

subbing into denominator

$$\therefore \frac{dy}{dt} = \boxed{\frac{0.4e^{0.4t}}{1+e^{0.8t}} \text{ or } \frac{2(e^{0.4t})}{5(1+e^{0.8t})}}$$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Question 6 continued

(c) question is asking us to solve this 1O.D.E in part (a)

$$\frac{dV}{dt} = 3 - \frac{4}{1+e^{0.8t}} - 0.4V$$

rearrange it so in form:

$$\frac{dy}{dx} + Py = Q$$

$$\frac{dV}{dt} + 0.4V = 3 - \frac{4}{1+e^{0.8t}}$$

Straight away can see that we cannot solve this 1 O.D.E by separation of variables as involves addition/subtraction rather than the product of 2 variables and $\frac{dy}{dx}$

· next see if can use reverse product rule

...consider LHS:

$$\frac{dV}{dt} + 0.4V \qquad \frac{d}{dt}(V) = \frac{dV}{dt} \checkmark \qquad \frac{d}{dt}(1) \neq 0.4$$

$\therefore$ not reverse product rule

$\therefore$ only way is through introducing an integration factor: $I.F = e^{\int P\,dt}$

$$e^{\int 0.4\,dt} = e^{0.4t}$$

multiplying through by $e^{0.4t}$

$$e^{0.4t}\frac{dV}{dt} + e^{0.4t}(0.4)V = e^{0.4t}\left(3 - \frac{4}{1+e^{0.8t}}\right)$$

$$\frac{d}{dt}(V) = \frac{dV}{dt} \checkmark \qquad \frac{d}{dt}(e^{0.4t}) = 0.4e^{0.4t} \checkmark$$

now can check if reverse product rule (above)

$\therefore$ CAN rewrite equation as derivative of product of $e^{0.4t}$ and V

$$\frac{d}{dt}\left(e^{0.4t}V\right) = 3e^{0.4t} - \frac{4e^{0.4t}}{1+e^{0.8t}}$$

integrating both sides and solving for V

$$Ve^{0.4t} = \int \left(3e^{0.4t} - \frac{4e^{0.4t}}{1+e^{0.8t}}\right) dt$$

$$\Rightarrow Ve^{0.4t} = \int \left(3e^{0.4t} - 10 \times \left(\frac{0.4e^{0.4t}}{1+e^{0.8t}}\right)\right) dt$$

recognising answer to (b)

$$\frac{d}{dt}\left(\arctan(e^{0.4t})\right) = \frac{0.4e^{0.4t}}{1+e^{0.8t}}$$

$$\int \frac{0.4e^{0.4t}}{1+e^{0.8t}} = \arctan(e^{0.4t}) + c$$

Question 6 continued

G.S: $Ve^{0.4t} = \frac{3}{0.4}e^{0.4t} - 10\arctan(e^{0.4t}) + c$

$Ve^{0.4t} = \frac{15}{2}e^{0.4t} - 10\arctan(e^{0.4t}) + c$

now to represent a particular solution need to sub in initial conditions

at $t=0, V=0$

$10e^0 = \frac{15}{2}e^0 - 10\arctan(e^0) + c$

$10 = \frac{15}{2} - \frac{5}{2}\pi + c$

$\frac{5}{2} + \frac{5}{2}\pi = c$

$\Rightarrow c = \frac{5(1+\pi)}{2}$

$\therefore Ve^{0.4t} = \frac{15}{2}e^{0.4t} - 10\arctan(e^{0.4t}) + 5\left(\frac{1+\pi}{2}\right)$

$\div e^{0.4t}$

$V = \frac{15}{2} - \frac{10}{e^{0.4t}}\arctan(e^{0.4t}) + \frac{5(\pi+1)}{2e^{0.4t}}$

using log laws on denominators

$\boxed{V = \frac{15}{2} - 10e^{-0.4t}\arctan(e^{0.4t}) + \frac{5}{2}(\pi+1)e^{-0.4t}}$

(d) sub in $t=10$ into above expression for V:

$V = 7.5 - 10e^{-0.4(10)}\arctan(e^{0.4(10)}) + 2.5(\pi+1)e^{-0.4(10)}$

on calc.

$= 7.40529\ldots$

working out % error

$\frac{7.40529\ldots - 8}{8} \times 100$

$\Rightarrow$ model predicts 7.43...% below actual value
$\therefore$ model not very accurate

(Total for Question 6 is 12 marks)

7. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

(a) Explain why, for $n \in \mathbb{N}$

$$\sum_{r=1}^{2n} (-1)^r \mathrm{f}(r) = \sum_{r=1}^{n} \left(\mathrm{f}(2r) - \mathrm{f}(2r-1)\right)$$

for any function $\mathrm{f}(r)$.

(2)

(b) Use the standard summation formulae to show that, for $n \in \mathbb{N}$

$$\sum_{r=1}^{2n} r\left((-1)^r + 2r\right)^2 = n(2n+1)(8n^2+4n+5)$$

(6)

(c) Hence evaluate

$$\sum_{r=14}^{50} r\left((-1)^r + 2r\right)^2$$

(4)

(a) when r is even, $(-1)^r = 1$

when r is odd, $(-1)^r = 1$

$\therefore \sum_{r=1}^{2n} (-1)^r f(r)$ summation must alternate between +ve and -ve terms

↳ seen also in Maclaurin series for $\sin x$

where $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots (-1)^r \frac{x^{2r}}{(2r)!}$

now the integers from 1 to 2n : $\sum_{r=1}^{2n} (-1)^r f(r) = -f(1) + f(2) - f(3) + f(4) - \dots - f(2n-1) + f(2n)$

can be split into:

* even integers of the form $2r$ from 1 to n (added)
* odd integers of the form $2r-1$ from 1 to n (substracted)

$= \underbrace{f(2) + f(4) + \dots + f(2n)}_{\sum_{r=1}^{n} f(2r)} - \left(\underbrace{f(1) + f(3) + \dots + f(2n-1)}_{\sum_{r=1}^{n} f(2r-1)}\right)$

Question 7 continued

$\therefore$ rewritten as

$$\sum_{r=1}^{n} f(2r) - \sum_{r=1}^{n} (2r-1)$$

(b) $\sum_{r=1}^{2n} r((-1)^r+2r)^2$ expand brackets

$$= \sum_{r=1}^{2n} r((-1)^{2r} + 4r(-1)^r + 4r^2)$$

$$= \sum_{r=1}^{2n} (r + 4r^2(-1)^r + 4r^3)$$

Splitting to use standard summation formulae:

- Sum of 'n' natural no.s : $\sum 1 = 1$
- Sum of 'n' constant terms: $\sum r = \frac{1}{2}n(n+1)$
- sum of squares: try get (a) $\sum r^2 = \frac{1}{6}n(n+1)(2n+1)$
- Sum of cubes: $\sum r^3 = \frac{1}{4}n^2(n+1)^2$

$$\Rightarrow \sum_{r=1}^{2n}(4r^3+r) + \sum_{r=1}^{2n}(4r^2(-1)^r) = \sum_{r=1}^{2n}(4r^3+r) + 4\sum_{r=1}^{2n}(r^2(-1)^r)$$

replacing with $f(2r)-f(2r-1)$ from (a)

$$= \sum_{r=1}^{2n}(4r^3+r) + 4\sum_{r=1}^{n}((2r)^2-(2r-1)^2)$$

expand

$$= \sum_{r=1}^{2n}(4r^3+r) + 4\sum_{r=1}^{n}(4r^2-(4r^2-4r+1))$$

$$= \sum_{r=1}^{2n}(4r^3+r) + 4\sum_{r=1}^{n}(4r-1)$$

$$= \sum_{r=1}^{2n}(4r^3+r) + \sum_{r=1}^{n}(16r-4)$$

$$= 4\sum_{r=1}^{2n} r^3 + 4\sum_{r=1}^{2n} r + 16\sum_{r=1}^{n} r - 4\sum_{r=1}^{n} 1$$

$$= 4\left(\frac{1}{4}(2n)^2(2n+1)^2\right) + 4\left(\frac{1}{2}(2n)(2n+1)\right) + 16\left(\frac{1}{2}n(2n+1)\right) - 4n$$

WAY 1 : simplify fractions

$$(2n)^2(2n+1)^2 + n(2n+1) + 4n(2(n+1)-1)$$

expand end bracket

$$= (2n)^2(2n+1)^2 + n(2n+1) + 4n(2n+1)$$

factorise n and $2n+1$ out

$$= n(2n+1)[4n(2n+1)+1+4]$$

$$= \boxed{n(2n+1)(8n^2+4n+5)}$$

WAY 2: factorise n out

$n(4n(2n+1)^2+(2n+1)+8(n+1)-4)$

expand brackets

$n(4n(4n^2+2n+1)+2n+1+8n+8-4)$

$n(16n^3+16n^2+4n+2n+1+8n+8-4)$

collect like terms

$n(16n^3+16n^2+14n+5)$

WAY 1: then 'working backward' see $(2n+1)(8n^2+4n+5)$

expand to $16n^3+16n^2+14n+5$ i.e cubic

$\boxed{=n(2n+1)(8n^2+4n+5)}$

WAY 2: or could've tried

$$\begin{array}{r|l} & 8n^2+4n+5 \\ \hline 2n+1 & 16n^3+16n^2+14n+5 \\ & -16n^3+8n^2 \\ \hline & 8n^2+14n \\ & -8n^2+4n \\ \hline & 10n+5 \\ & -10n+5 \\ \hline & 0 \end{array}$$

OR

WAY 3: by inspection

$(2n+1)(an^2+bn+c)=16n^3+16n^2+14n+5$

$\dots n^3$:

$\div 2 \quad 2a=16 \quad \div 2$

$\Rightarrow a=8$

$\dots n^2$:

$a+2b=16$

"8" $+2b=16$

$\div 2 \quad 2b=8 \quad \div 2$

$\Rightarrow b=4$

$\dots$ constant:

$c=5$

$\boxed{\therefore n(2n+1)(8n^2+4n+5)}$

(c) knowing formula for when $r\neq 1$

$$\sum_{r=k}^{n}=\sum_{r=1}^{n}-\sum_{r=1}^{k-1}$$

$$\sum_{r=14}^{50} r((-1)^r+2r)^2=\sum_{r=1}^{50} r((-1)^r+2r)^2-\sum_{r=1}^{13} r((-1)^r+2r)^2$$

↳ want to use (b)

but $2n=50$

$\therefore$ sub in $n=15$ into above

Question 7 continued

$= 25(51)(8\times(25)^2 + 4(25) + 5)$

but for $\sum_{r=1}^{13}$ $\div 2 \; 2n = 13 \div 2$

$\Rightarrow n = 6.5$ but summation relies on $n \in \mathbb{R}$

to subtract series up to 13

let's $-\sum_{r=1}^{12} r((-1)^r + 2r)^2$ and a further 13 term: $-13((-1)^{13}+2(13))^2$

i.e $n=6$

$$\sum_{r=14}^{50} r((-1)^r+2r)^2 = 25(51)((8\times25^2)+4(25)+5)) - 6(13)((8\times6^2))+4(6)+5) - 13((-1)^{13}+2(13))^2$$

$= 6{,}508{,}887 - 24{,}726 - 8{,}125$

$\boxed{= 6{,}476{,}024}$

(Total for Question 7 is 12 marks)

P 7 2 7 9 4 A 0 2 3 2 8

8. A colony of small mammals is being studied.
In the study, the mammals are divided into 3 categories

N (newborns)	0 to less than 1 month old
J (juveniles)	1 to 3 months old
B (breeders)	over 3 months old

(a) State one limitation of the model regarding the division into these categories.

(1)

A model for the population of the colony is given by the matrix equation

$$\begin{pmatrix} N_{n+1} \\ J_{n+1} \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ a & b & 0 \\ 0 & 0.48 & 0.96 \end{pmatrix} \begin{pmatrix} N_n \\ J_n \\ B_n \end{pmatrix}$$

where a and b are constants, and N_n, J_n and B_n are the respective numbers of the mammals in each category n months after the start of the study.

At the start of the study the colony has breeders only, with no newborns or juveniles.

According to the model, after 2 months the number of newborns is 48 and the number of juveniles is 40

(b) (i) Determine the number of mammals in the colony at the start of the study.

(ii) Show that $a = 0.8$

(4)

(c) Determine, in terms of b,

$$\begin{pmatrix} 0 & 0 & 2 \\ 0.8 & b & 0 \\ 0 & 0.48 & 0.96 \end{pmatrix}^{-1}$$

(3)

Given that the model predicts approximately 1015 mammals **in total** at the start of a particular month, and approximately 596 **newborns**, 464 **juveniles** and 437 **breeders** at the start of the next month,

(d) determine the value of b, giving your answer to 2 decimal places.

(3)

It is decided to monitor the number of **newborn** males and females as a part of the study. Assuming that 42% of newborns are male,

(e) refine the matrix equation for the model to reflect this information, giving a reason for your answer.
(*There is no need to estimate any unknown values for the refined model, but any known values should be made clear.*)

(2)

Question 8 continued

(a)... possible reasons:

- 1 month is too old for a 'newborn'
- mammals may not start breeding at exactly 3 months old
- being >3mnths old doesn't necessarily mean can breed
- some juveniles may be breeders

(b)(i) METHOD 1: using given matrix equation for 1 month, THEN 2

let no. of mammals at start $= x$

$$\begin{pmatrix} N_0 \\ J_0 \\ B_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix}$$

but here given matrix equation for 'n' months after start ∴ subbing $n=0$ into equation get one for 1 month after start study

$$\begin{pmatrix} N_1 \\ J_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ a & b & 0 \\ 0 & 0.48 & 0.96 \end{pmatrix}\begin{pmatrix} N_0 \\ J_0 \\ B_0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 2 \\ a & b & 0 \\ 0 & 0.48 & 0.96 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix}$$

evaluate this using matrix multiplication "rows into columns"

let product matrix be $\begin{pmatrix} A(1,1) \\ A(2,1) \\ A(3,1) \end{pmatrix}$

...for $A(1,1)$:
$(0)(0) + 0 + 2x$
$= 2x$

...for $A(2,1)$:
$a(0) + b(0) + 0$
$= 0$

...for $A(3,1)$:
$0 + (0.48)(0) + 0.96x$
$= 0.96x$

$$\therefore \begin{pmatrix} 2x \\ 0 \\ 0.96x \end{pmatrix}$$

but we only have information on no. of months after 2 months

∴ subbing $n=1$ into vector equation

$$\begin{pmatrix} N_2 \\ J_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ a & b & 0 \\ 0 & 0.48 & 0.96 \end{pmatrix}\begin{pmatrix} N_1 \\ J_1 \\ B_1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 2 \\ a & b & 0 \\ 0 & 0.48 & 0.96 \end{pmatrix}\begin{pmatrix} 2x \\ 0 \\ 0.96x \end{pmatrix}$$

evaluating matrix multiplication "rows into columns"

let product matrix $= \begin{pmatrix} A(1,0) \\ A(2,0) \\ A(3,0) \end{pmatrix}$

...for $A(1,1)$: $0(2x) + 0(0) + 2(0.96x)$
$= 1.92x$

Question 8 continued

...for $A(2,1)$: $a(2x) + b(0) + 0(0.96x)$
$= 2ax$

...for $A(3,1)$: $0(2x) + 0(0.48) + 0.96(0.96x)$
$= 0.921x$

and equating to info from question – newborns

$1.92x = 48$
$\div 1.92 \quad \div 1.92$
$x = 25$

$\boxed{\therefore 25 \text{ mammals}}$

METHOD 2: squaring matrix for month 2

$$\begin{pmatrix} N_2 \\ J_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ a & b & 0 \\ 0 & 0.48 & 0.96 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ a & b & 0 \\ 0 & 0.48 & 0.96 \end{pmatrix} \begin{pmatrix} N_0 = 0 \\ J_0 = 0 \\ B_0 = x \end{pmatrix}$$

$$= \begin{pmatrix} 0+0+0 & 0+0+2(0.48) & 0+0+2(0.96) \\ 0+ab+0 & 0+b^2+0 & 2a+0+0 \\ 0+0.48a+0 & 0+0.48b+(0.96)(0.48) & 0+0+(0.96)^2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0.96 & 1.92 \\ ab & b^2 & 2a \\ 0.48a & 0.48a + 0.4603 & 0.9216 \end{pmatrix}$$

$$\begin{pmatrix} N_2 \\ J_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} 0 & 0.96 & 1.92 \\ ab & b^2 & 2a \\ 0.48a & 0.48b + 0.4603 & 0.96^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix}$$

matrix multiplication

$$= \begin{pmatrix} 1.92x \\ 2ax \\ 0.9216x \end{pmatrix}$$

and equate to question

$48 = 1.92x$
$\div 1.92 \quad \div 1.92$
$x = 25$

$\boxed{\therefore 25 \text{ mammals}}$

(ii) now using information for juveniles:

$2ax = 40$

$2a\text{"}25\text{"} = 40$
$\div 50 \quad \div 50$

$\Rightarrow a = 4/5$

$\boxed{= 0.8}$

(c) find INVERSE of a 3x3 matrix - MANUALLY due to unknown 'b' ∴ following steps:-

Let $A = \begin{pmatrix} 0 & 0 & 2 \\ 0.8 & b & 0 \\ 0 & 0.48 & 0.96 \end{pmatrix}$

step 1: find det(A)

$= 0\begin{vmatrix} b & 0 \\ 0.48 & 0.96 \end{vmatrix} - 0\begin{vmatrix} 0.8 & 0 \\ 0 & 0.96 \end{vmatrix} + 2\begin{vmatrix} 0.8 & b \\ 0 & 0.48 \end{vmatrix}$

$= 0 - 0 + 2(0.384)$

$= 0.768$

$\therefore \frac{1}{\det(A)} = \frac{1}{0.768} = \frac{125}{96}$

step 2: find matrix of minors - det. of 2x2 matrix left after deleting all rows and columns corresponding to given element

$M = \begin{pmatrix} 0.96b & 0.8(0.96)-0 & 0.8(0.48)-0 \\ -0.96 & 0(0.96)-2(0) & 0(0.48)-0 \\ 0-2b & 0-2(0.8) & 0(b)-0.8(0) \end{pmatrix}$

$= \begin{pmatrix} 0.96b & 0.768 & 0.384 \\ -0.96 & 0 & 0 \\ -2b & -1.6 & 0 \end{pmatrix}$

step 3: matrix of cofactors - change sign of '-ve'

$\begin{pmatrix} \overset{+}{a} & \overset{-}{b} & \overset{+}{c} \\ \overset{-}{d} & \overset{+}{e} & \overset{-}{f} \\ \overset{+}{g} & \overset{-}{h} & \overset{+}{i} \end{pmatrix}$ $\begin{pmatrix} 0.96b & -0.768 & 0.384 \\ 0.96 & 0 & -0 \\ -2b & 1.6 & 0 \end{pmatrix}$

step 4: transpose i.e switch places of highlighted

$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ $\therefore C^T = \begin{pmatrix} 0.96b & 0.96 & -2b \\ -0.768 & 0 & 1.6 \\ 0.384 & 0 & 0 \end{pmatrix}$

step 5: $A^{-1} = \frac{1}{\det(A)} C^T$

$= \frac{125}{96}\begin{pmatrix} 0.96b & 0.96 & -2b \\ -0.768 & 0 & 1.6 \\ 0.384 & 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1.25b & 1.25 & -2.6042b \\ -1 & 0 & 2.80333 \\ 0.5 & 0 & 0 \end{pmatrix}$

(d) METHOD 1: using vector equation

subbing new info into matrix equation

$$\begin{pmatrix}596\\464\\437\end{pmatrix} = \begin{pmatrix}0 & 0 & 2\\ a & b & 0\\ 0 & 0.48 & 0.96\end{pmatrix}\begin{pmatrix}N_\lambda\\ J_\lambda\\ B_\lambda\end{pmatrix}$$

but given info on beginning of the month ∴ need $\begin{pmatrix}N_\lambda\\ J_\lambda\\ B_\lambda\end{pmatrix}$

$$\therefore \begin{pmatrix}N_\lambda\\ J_\lambda\\ B_\lambda\end{pmatrix} = \underbrace{\begin{pmatrix}0 & 0 & 2\\ 0.8 & b & 0\\ 0 & 0.48 & 0.96\end{pmatrix}^{-1}}_{\text{part (c)}}\begin{pmatrix}596\\464\\437\end{pmatrix}$$

$$\begin{pmatrix}N_\lambda\\ J_\lambda\\ B_\lambda\end{pmatrix} = \begin{pmatrix}1.25b & 1.25 & -2.60426\\ -1 & 0 & 2.80333\\ 0.5 & 0 & 0\end{pmatrix}\begin{pmatrix}596\\464\\437\end{pmatrix}$$

∴ matrix multiplication: "rows into columns"; let product matrix be $\begin{pmatrix}A(1,1)\\A(2,1)\\A(3,1)\end{pmatrix}$

∴ for $A(1,1)$:

$1.25(b)(596) + 1.25(464) - 2.60426(437)$

$= -393.0345b + 580$

∴ for $A(2,1)$:

$-1(596 + 0(464) + 437(2.80333)$

$= 629.05521$

∴ for $A(3,1)$: $0.5(596) + 0(464) + 0(437)$

$= 299$

$$\begin{pmatrix}580 - 393.035b\\ 314.416.\\ 298\end{pmatrix}$$

∴ sum = 1015

$580 - 393.0356 + 314.4166.. + 298 = 1015$

$\Rightarrow \boxed{b = 0.45}$

METHOD 2: uses formulations of matrix equations

let x, y, z = newborns, juveniles and breeders

using matrix in (c)

$$\begin{pmatrix}0 & 0 & 2\\ 0.8 & b & 0\\ 0 & 0.48 & 0.96\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}596\\464\\437\end{pmatrix}$$

matrix multiplication

$\div 2 \quad 2z = 596 \quad \div 2 \quad -①$

$z = 298$

$\Rightarrow 0.8x + by = 464 \quad -②$

$\Rightarrow 0.48y + 0.96z = 437 \quad ③$

sub z into (3) for 'y'

$0.48y = 437 - 0.96(298)$

$\div 0.48 \quad = 150.92 \quad \div 0.48$

$\Rightarrow y = 3773/12$

sub 'y' into total equation

$$x + y + z = 1015$$

$$x + \frac{3773}{12} + 298 = 1015$$

$$\Rightarrow x = \frac{4831}{12}$$

and into ② for 'b':

$$0.8\left(\frac{4831}{12}\right) + b\left(\frac{3773}{12}\right) = 466$$

$$\frac{3773}{12} b = \frac{15{,}476}{75}$$

$$\Rightarrow b = \boxed{0.45 \ (2 \text{ d.p})}$$

(e) get original equation first

$$\begin{pmatrix} N_{n+1} \\ J_{n+1} \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ a & 0.45 & 0 \\ 0 & 0.48 & 0.96 \end{pmatrix} \begin{pmatrix} N_n \\ J_n \\ B_n \end{pmatrix}$$

now splitting newborns into NF_{n+1} and NM_{n+1}

so $\begin{pmatrix} N_{n+1} \\ J_{n+1} \\ B_{n+1} \end{pmatrix} \rightarrow \begin{pmatrix} NF_{n+1} \\ NM_{n+1} \\ J_{n+1} \\ B_{n+1} \end{pmatrix}$

and $\begin{pmatrix} N_n \\ J_n \\ B_n \end{pmatrix} \rightarrow \begin{pmatrix} NF_n \\ NM_n \\ J_n \\ B_n \end{pmatrix}$

now manipulate main matrix A

- know need 4×4

...row 1:

- the $A_{(1,4)}$ needs to be the 42% of 2 to get just males:

$$\therefore 0.84$$

row 1: 0 0 0 0.84

...row 2:

now similarly for $A_{(1,4)}$ with 58% of 2

$$= 1.16 \text{ (to get females)}$$

...row 3:

notice 'a' but 'a' split into 0.42a males and 0.58 females

row 3: 0.42a 0.58a b 0

row 4: 0 0 0.48 0.96

$$\Rightarrow \begin{pmatrix} 0 & 0 & 2 \\ a & 0.45 & 0 \\ 0 & 0.48 & 0.96 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0.84 \\ 0 & 0 & 0 & 1.16 \\ 0.42a & 0.55a & b & 0 \\ 0 & 0 & 0.48 & 0.96 \end{pmatrix}$$

Question 8 continued

$$\therefore \begin{pmatrix} NM_{n+1} \\ NF_{n+1} \\ J_{n+1} \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0.84 \\ 0 & 0 & 0 & 1.16 \\ 0.42a & 0.58a & 6 & 0 \\ 0 & 0 & 0.48 & 0.96 \end{pmatrix} \begin{pmatrix} NF_n \\ NM_n \\ J_n \\ B_n \end{pmatrix}$$

(Total for Question 8 is 13 marks)

TOTAL FOR PAPER IS 75 MARKS